

Duration : 144 minutes



# Linear Algebra

## Exam

### Common part

### Fall 2015

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## Questions

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For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

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The notation and terminology of this exam are those used in the exercise sheets and the lectures of the course Linear Algebra given during the Fall semester 2015.

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## Notation

- For a matrix  $A$ ,  $a_{ij}$  denotes the entry of  $A$  in row  $i$  and column  $j$ .
- For a vector  $\mathbf{x}$ ,  $x_i$  denotes the  $i$ -th coordinate of  $\mathbf{x}$ .
- $I_m$  denotes the  $m \times m$  identity matrix.
- $\mathbb{P}_n$  is the vector space of polynomials of degree less than or equal to  $n$ .
- The inner product of vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  is defined as  $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$ .
- The length of a vector  $\mathbf{x} \in \mathbb{R}^n$  is defined as  $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ .

## First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

**Question 1 :** Let  $A$  be a  $5 \times 6$  matrix and let  $\mathbf{b} \in \mathbb{R}^5$  be a vector. The set of solutions of the matrix equation  $A\mathbf{x} = \mathbf{b}$  cannot be equal to

- ☐ a nonempty finite set
- ☐ a subspace of  $\mathbb{R}^6$  of dimension 1
- ☐ a subspace of  $\mathbb{R}^6$  of dimension 2
- ☐ the empty set

**Question 2 :** Consider a set  $\mathcal{S}$  of polynomials  $p_1, \dots, p_5$ , such that the degree of  $p_k$  equals  $k$  for  $k = 1, \dots, 5$ . Then

- ☐ one can obtain a basis of  $\mathbb{P}_5$  by adding the polynomial  $p(t) = t^5 - t$  to the set  $\mathcal{S}$
- ☐ one can extract a basis of  $\mathbb{P}_5$  from the set  $\mathcal{S}$
- ☐ the set  $\mathcal{S}$  forms a basis of  $\mathbb{P}_5$
- ☐ one can obtain a basis of  $\mathbb{P}_5$  by adding the polynomial  $p(t) = 5$  to the set  $\mathcal{S}$

**Question 3 :** Let  $A$  and  $B$  be two invertible matrices of the same size. If the matrices  $C$  and  $D$  are defined by  $C = AB$  and  $D = A^T + B^T$ , then

- ☐  $D$  is always invertible, but  $C$  is not necessarily invertible
- ☐  $C$  is not necessarily invertible, and  $D$  is not necessarily invertible
- ☐  $C$  and  $D$  are always invertible
- ☐  $C$  is always invertible, but  $D$  is not necessarily invertible

**Question 4 :** Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2h - 4 \\ -3 - h \end{pmatrix},$$

where  $h$  is a real parameter. Then the matrix equation  $A\mathbf{x} = \mathbf{b}$

- ☐ has infinitely many solutions for  $h = 3$
- ☐ has infinitely many solutions for  $h = -3$
- ☐ has finitely many solutions for every  $h \in \mathbb{R}$
- ☐ has infinitely many solutions for every  $h \neq \pm 3$

**Question 5 :** Let  $A$  be an  $n \times n$  matrix. Three of the following four statements are equivalent to each other. Which statement is not one of those three?

- ☐ The set of columns of  $A$  forms an orthonormal basis of  $\mathbb{R}^n$
- ☐  $\det A = 1$
- ☐ The set of rows of  $A$  forms an orthonormal basis of  $\mathbb{R}^n$
- ☐  $AA^T = I_n$

**Question 6 :** Let  $B$  be an  $m \times n$  matrix such that  $BB^T = I_m$ . Then

- ☐ the columns of  $B$  form an orthonormal set
- ☐ the rows of  $B$  form an orthonormal set
- ☐  $B^TB = I_n$
- ☐  $B$  is invertible

**Question 7 :** Let  $A = \begin{pmatrix} 7 & -24 \\ -24 & -7 \end{pmatrix}$ . There is an orthogonal diagonalization  $A = PDP^T$ , where  $D$  is a diagonal matrix, and where

- |  |  |
|--|--|
| <input type="checkbox"/> $P = \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix}$ | <input type="checkbox"/> $P = \begin{pmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{pmatrix}$ |
| <input type="checkbox"/> $P = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$ | <input type="checkbox"/> $P = \begin{pmatrix} 12/13 & 5/13 \\ 5/13 & -12/13 \end{pmatrix}$ |

**Question 8 :** There exists a polynomial  $p(t) = a_0 + a_1t + a_2t^2$  with real coefficients  $a_0, a_1, a_2$  such that

$$p(-1) = 1, \quad p(0) = 1, \quad p(1) = 3, \quad p(2) = b,$$

- ☐ for one single real value of  $b$
- ☐ for no real value of  $b$
- ☐ for every real value of  $b$
- ☐ for a finite set of at least two real values of  $b$

**Question 9 :** The area of the triangle determined by the three points

$$(0, 0), \quad (-1, 5), \quad (3, -1)$$

equals

- ☐ 7
- ☐ 8
- ☐ 16
- ☐ 14

**Question 10 :** Let

$$\mathbf{x} = \begin{pmatrix} -1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{et} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}.$$

Then the orthogonal projection of  $\mathbf{x}$  on the subspace  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  is:

☐  $\begin{pmatrix} -1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$ 
☐  $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$ 
☐  $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ 
☐  $\begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \end{pmatrix}$

**Question 11 :** The value of the parameter  $b \in \mathbb{R}$ , for which the vector  $\mathbf{w} = \begin{pmatrix} 4 \\ b \\ 0 \end{pmatrix}$  lies in

the plane in  $\mathbb{R}^3$  spanned by the vectors  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$  and  $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ , is:

- ☐  $b = 2$   
☐  $b = -1$   
☐  $b = -5$   
☐  $b = -2$

**Question 12 :** Let  $A$  be an  $m \times n$  matrix,  $\mathbf{b}$  a vector in  $\mathbb{R}^m$ , and  $\hat{\mathbf{b}}$  the orthogonal projection of  $\mathbf{b}$  on  $\text{Col}(A)$ . Then

- ☐ the least-squares solution of  $A\mathbf{x} = \mathbf{b}$  is  $A^{-1}\hat{\mathbf{b}}$   
☐ the matrix  $A^T A$  is invertible  
☐ every solution of  $A^T A\mathbf{x} = A^T \mathbf{b}$  is a least-squares solution of  $A\mathbf{x} = \mathbf{b}$   
☐ the equation  $A\mathbf{x} = \hat{\mathbf{b}}$  has a unique solution

**Question 13 :** The set of symmetric  $2 \times 2$  matrices forms a vector space with the usual rules for adding matrices and for multiplying a matrix by a scalar. Let

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 1 \end{pmatrix} \quad \text{be a matrix and let} \quad \mathcal{B} = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be a basis of this vector space. If we represent  $A$  in the basis  $\mathcal{B}$ , then the coefficient of the third vector of  $\mathcal{B}$  is

- ☐ 9  
☐ -3  
☐ 2  
☐ 1

**Question 14 :** The determinant of the matrix

$$\begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 5 & 4 & 3 & -2 \end{pmatrix}$$

equals

- ☐ 24
- ☐ 12
- ☐ -12
- ☐ -24

**Question 15 :** Let  $A$  and  $B$  be  $n \times n$  matrices, with  $B$  invertible. Suppose that  $\lambda$  is an eigenvalue of  $A$ , and also an eigenvalue of  $B$ .

Among the following statements:

- (a)  $\lambda$  is an eigenvalue of the matrix  $A + B$ ,
- (b)  $\lambda$  is an eigenvalue of the matrix  $AB$ ,
- (c)  $\lambda$  is an eigenvalue of the matrix  $BAB^{-1}$ ,
- (d)  $\lambda^2$  is an eigenvalue of the matrix  $BA$ ,

which are always true?

- ☐ only (c)
- ☐ (a), (c), and (d)
- ☐ only (d)
- ☐ (a) and (b)

**Question 16 :** Let

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 5 & 0 & 1 \\ -2 & 11 & 0 \end{pmatrix}.$$

Compute the  $LU$  factorization of  $A$  (using only the elementary row operation that adds a multiple of one row to another row below it). Then the entry  $\ell_{31}$  of  $L$  equals

- ☐ 2
- ☐ 3
- ☐ -1
- ☐ -2

**Question 17 :** Let

$$A = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 3 & -3 \\ 2 & 6 & -4 \end{pmatrix}.$$

If  $B = A^{-1}$ , then the entry  $b_{11}$  of  $B$  equals

☐  $-1$

☐  $-\frac{1}{5}$

☐  $\frac{1}{5}$

☐  $\frac{1}{3}$

**Question 18 :** Consider the matrix  $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$  and the vector  $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \end{pmatrix}$ .

Then the least-squares solution  $\hat{\mathbf{x}} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}$  of the equation  $A\mathbf{x} = \mathbf{b}$  satisfies

☐  $\hat{x}_2 = 3$

☐  $\hat{x}_2 = -3$

☐  $\hat{x}_2 = 4$

☐  $\hat{x}_2 = -4$

**Question 19 :** Consider the transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{P}_2$  defined by the formula

$$T\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = (c - b) + (a - 2b + c)t + (a - b)t^2.$$

Then

☐  $T$  is linear and its rank equals 1

☐  $T$  is linear and its rank equals 2

☐  $T$  is linear and one-to-one

☐  $T$  is not linear

**Question 20 :** Let  $h$  be a real parameter. The vectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ h \\ 1 \end{pmatrix}$$

form a basis of  $\mathbb{R}^4$

☐ only if  $h = 0$

☐ only if  $h \neq \frac{1}{2}$

☐ only if  $h = \frac{1}{2}$

☐ only if  $h \neq 0$

**Question 21 :** The system of linear equations

$$\begin{cases} -x_1 & -2x_3 & = 0 \\ & x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \\ & -2x_2 - 2x_3 - 3x_4 = 0 \end{cases}$$

- ☐ does not have a free variable
- ☐ has exactly two free variables
- ☐ has exactly one free variable
- ☐ has exactly three free variables

**Question 22 :** Let

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

Then

- ☐ the only eigenvalue of  $A$  is 2
- ☐ the eigenvalues of  $A$  are 0, 2, and  $-2$
- ☐ the eigenvalues of  $A$  are 1,  $-1$ , 2, and  $-2$
- ☐ the eigenvalues of  $A$  are 2 and  $-2$

**Question 23 :** Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 7 & -2 \end{pmatrix}.$$

Then

- ☐  $A$  has three distinct eigenvalues
- ☐  $A$  is orthogonally diagonalizable
- ☐  $A$  is diagonalizable but not orthogonally diagonalizable
- ☐  $A$  is not diagonalizable

**Question 24 :** Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear transformation defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 4x_3 \\ 3x_1 + 5x_2 - 2x_3 \\ x_1 + x_2 + 4x_3 \end{pmatrix}. \text{ Let } \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Let  $M$  be the matrix that represents  $T$  in the basis  $\mathcal{B}$ ; in other words, we have  $[T(\mathbf{x})]_{\mathcal{B}} = M[\mathbf{x}]_{\mathcal{B}}$  for every  $\mathbf{x} \in \mathbb{R}^3$ . Then

- |   |   |
|---|---|
| <input type="checkbox"/> $M = \begin{pmatrix} 4 & 6 & 6 \\ 0 & 8 & 2 \\ 0 & 3 & 1 \end{pmatrix}$    | <input type="checkbox"/> $M = \begin{pmatrix} 6 & 0 & -2 \\ 2 & 6 & -8 \\ 1 & 2 & -3 \end{pmatrix}$ |
| <input type="checkbox"/> $M = \begin{pmatrix} 6 & 2 & 1 \\ 0 & 6 & 2 \\ -2 & -8 & -3 \end{pmatrix}$ | <input type="checkbox"/> $M = \begin{pmatrix} 4 & 0 & 0 \\ 6 & 8 & 3 \\ 6 & 2 & 1 \end{pmatrix}$    |

## Second part: true/false questions

For each question, mark (without erasures) TRUE if the statement is **always true** and FALSE if it is **not always true** (i.e., it is sometimes false).

**Question 25 :** Let  $p \in \mathbb{P}_n$  be a polynomial and let  $p'$  be the derivative of  $p$ . The set  $\{p \in \mathbb{P}_n : p'(-1) \neq 0\}$  is a subspace of  $\mathbb{P}_n$ .

☐ TRUE      ☐ FALSE

**Question 26 :** Let  $A$  be an  $n \times n$  matrix and  $\lambda$  a non-zero real number. If the homogeneous system  $A\mathbf{x} = \mathbf{0}$  has exactly one solution, then the set of columns of  $(\lambda A)^T$  spans  $\mathbb{R}^n$ .

☐ TRUE      ☐ FALSE

**Question 27 :** If  $A$  and  $B$  are row equivalent matrices (in other words,  $B$  can be obtained from  $A$  using elementary row operations), then  $\text{Col}(A^T) = \text{Col}(B^T)$ .

☐ TRUE      ☐ FALSE

**Question 28 :** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation that is one-to-one. Let  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  be a linearly independent set of vectors in  $\mathbb{R}^n$ . Then  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_p)\}$  is a linearly independent set of vectors in  $\mathbb{R}^m$ .

☐ TRUE      ☐ FALSE

**Question 29 :** Let  $T : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  be a linear transformation, and let  $A$  be the matrix that represents  $T$  in the standard basis. If  $T$  is onto, then the null space of  $A$  has dimension 1.

☐ TRUE      ☐ FALSE

**Question 30 :** Let  $A$  be a  $7 \times 3$  matrix, whose first two columns are linearly independent, and whose third column equals its first. Then the matrix  $A^T A$  has size  $3 \times 3$ , is symmetric, and has rank 2.

☐ TRUE      ☐ FALSE

**Question 31 :** If  $A$  is a  $4 \times 4$  matrix of rank 1, and  $\lambda = 0$  is an eigenvalue of  $A$  with algebraic multiplicity 3, then  $A$  is diagonalizable. (The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial of the matrix.)

☐ TRUE      ☐ FALSE

**Question 32 :** Let  $A$  be an  $m \times n$  matrix. If  $\mathbf{x} \in \text{Col}(A)$  satisfies  $A^T \mathbf{x} = \mathbf{0}$ , then  $\|\mathbf{x}\| = 0$ .

☐ TRUE      ☐ FALSE